



22057208

**MATHEMATICS
HIGHER LEVEL
PAPER 2**

Wednesday 4 May 2005 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all **five** questions from this section.

1. [Maximum mark: 14]

- (a) Let \mathbf{R} be a rotation through k degrees, centre $(0, 0)$. \mathbf{R} maps the point $(5, 10)$ onto the point $(-2, 11)$. Find the matrix \mathbf{R} . [6 marks]

- (b) A transformation \mathbf{T} is represented by the matrix $\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$. Describe the geometric effect of applying transformation \mathbf{T} four times in succession. [2 marks]

- (c) The matrix \mathbf{Q} represents rotation \mathbf{R} followed by transformation \mathbf{T} . Show that

$$\mathbf{Q} = \begin{pmatrix} 2 & 1 \\ -1.4 & -0.2 \end{pmatrix}. \quad [1 \text{ mark}]$$

- (d) Under transformation \mathbf{Q} , find

- (i) the set of points which are mapped onto themselves;
- (ii) the image of the line $y = -x$. [5 marks]

2. [Maximum mark: 13]

The function f is defined by $f(x) = e^{px}(x+1)$, where $p \in \mathbb{R}$.

- (a) (i) Show that $f'(x) = e^{px}(p(x+1) + 1)$.
- (ii) Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x , n times. Use mathematical induction to prove that

$$f^{(n)}(x) = p^{n-1}e^{px}(p(x+1) + n), \quad n \in \mathbb{Z}^+. \quad [7 \text{ marks}]$$

- (b) When $p = \sqrt{3}$, there is a minimum point and a point of inflexion on the graph of f . Find the **exact** value of the x -coordinate of

- (i) the minimum point;
- (ii) the point of inflexion. [4 marks]

- (c) Let $p = \frac{1}{2}$. Let R be the region enclosed by the curve, the x -axis and the lines $x = -2$ and $x = 2$. Find the area of R . [2 marks]

3. [Maximum mark: 12]

(a) The plane π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -9 \end{pmatrix}$.

The plane π_2 has the equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

- (i) For points which lie in π_1 and π_2 , show that, $\lambda = \mu$.
- (ii) Hence, or otherwise, find a vector equation of the line of intersection of π_1 and π_2 . [5 marks]

- (b) The plane π_3 contains the line $\frac{2-x}{3} = \frac{y}{-4} = z+1$ and is perpendicular to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find the cartesian equation of π_3 . [4 marks]

- (c) Find the intersection of π_1 , π_2 and π_3 . [3 marks]

4. [Maximum mark: 16]

A company buys 44 % of its stock of bolts from manufacturer A and the rest from manufacturer B. The diameters of the bolts produced by each manufacturer follow a normal distribution with a standard deviation of 0.16 mm.

The mean diameter of the bolts produced by manufacturer A is 1.56 mm.

24.2 % of the bolts produced by manufacturer B have a diameter less than 1.52 mm.

- (a) Find the mean diameter of the bolts produced by manufacturer B. [3 marks]

A bolt is chosen at random from the company's stock.

- (b) Show that the probability that the diameter is less than 1.52 mm is 0.312, to three significant figures. [4 marks]

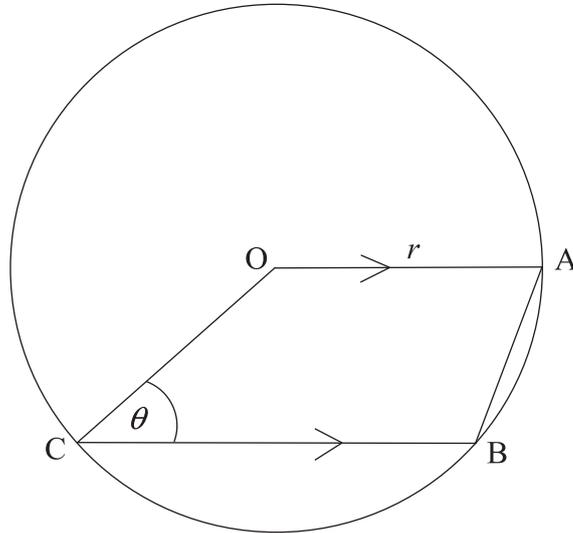
- (c) The diameter of the bolt is found to be less than 1.52 mm. Find the probability that the bolt was produced by manufacturer B. [3 marks]

- (d) Manufacturer B makes 8000 bolts in one day. It makes a profit of \$ 1.50 on each bolt sold, on condition that its diameter measures between 1.52 mm and 1.83 mm. Bolts whose diameters measure less than 1.52 mm must be discarded at a loss of \$ 0.85 per bolt. Bolts whose diameters measure over 1.83 mm are sold at a reduced profit of \$ 0.50 per bolt.

Find the expected profit for manufacturer B. [6 marks]

5. [Maximum mark: 15]

The diagram shows a trapezium OABC in which OA is parallel to CB. O is the centre of a circle radius r cm. A, B and C are on its circumference. Angle $OCB = \theta$.



Let T denote the area of the trapezium OABC.

- (a) Show that $T = \frac{r^2}{2}(\sin \theta + \sin 2\theta)$. [4 marks]

For a **fixed** value of r , the value of T varies as the value of θ varies.

- (b) Show that T takes its maximum value when θ satisfies the equation $4\cos^2 \theta + \cos \theta - 2 = 0$, and verify that this value of T is a maximum. [5 marks]
- (c) Given that the perimeter of the trapezium is 75 cm, find the maximum value of T . [6 marks]

SECTION B

Answer **one** question from this section.

Statistics

6. [Maximum mark: 30]

(i) Let X be a random variable with a Poisson distribution, such that $P(X > 2) = 0.404$. Find $P(X < 2)$. [4 marks]

(ii) An urn contains a large number of black and white balls. It is claimed that two-thirds $\left(\frac{2}{3}\right)$ of the balls are white. To check this claim, five balls are taken at random from the urn and the number of white balls recorded. This experiment is repeated 243 times, with the following results:

Number of white balls in the sample	0	1	2	3	4	5
Number of times this sample occurred	8	9	52	78	70	26

At the 5 % level of significance can the claim be accepted? [8 marks]

(iii) Peter keeps two ducks Ann and Bet, of different breeds. He uses a test to determine whether or not the eggs laid by each duck have, on average, the same weight. He weighs the next 12 eggs laid by each duck, and obtains the following results, where the weight is in grams.

Ann	63.1	63.6	65.3	65.7	62.0	64.8	64.3	63.2	64.9	66.6	64.1	62.3
Bet	66.8	66.9	64.1	64.0	65.8	63.6	67.2	66.4	67.3	65.0	67.3	65.1

It may be assumed that the weights are normally distributed, with equal standard deviation 2 grams.

(a) Write down suitable hypotheses for the test required. [1 mark]

(b) (i) Calculate an appropriate statistic to test the hypotheses.
 (ii) Using a 5 % level of significance, determine whether the eggs laid by each duck have the same mean weight. [6 marks]

(c) Find the maximum level of significance at which you would conclude that there is no difference between the mean weights of eggs laid by each duck. [2 marks]

(iv) The random variable X is normally distributed with mean μ . A random sample of 12 observations is taken on X , and it is found that

$$\sum_{i=1}^{12} (x_i - \bar{x})^2 = 99.$$

(a) Determine a 95 % confidence interval for μ . [5 marks]

(b) Another confidence interval [60.31, 65.69] is calculated for this sample. Find the confidence level for this interval. [4 marks]

Sets, Relations and Groups

7. [Maximum mark: 30]

(i) Define the operation # on the sets A and B by $A \# B = A' \cup B'$.

Show algebraically that

(a) $A \# A = A'$; [1 mark]

(b) $(A \# A) \# (B \# B) = A \cup B$; [2 marks]

(c) $(A \# B) \# (A \# B) = A \cap B$. [3 marks]

(ii) Let $S = \{\text{integers greater than 1}\}$. The relation R is defined on S by

$$m R n \Leftrightarrow \gcd(m, n) > 1, \text{ for } m, n \in S.$$

(a) Show that R is reflexive. [1 mark]

(b) Show that R is symmetric. [2 marks]

(c) Show using a counter example that R is not transitive. [3 marks]

(iii) Let $T = \{\text{all real numbers except 1}\}$. The operation $*$ is defined on T by

$$a * b = ab - a - b + 2, \text{ for } a, b \in T.$$

(a) Show that T is closed under the operation $*$. [5 marks]

You may now assume that T is a group under $*$.

(b) Find the identity element of T under $*$. [3 marks]

(c) (i) Prove by mathematical induction that

$$\overbrace{a * a * \dots * a}^{n \text{ times}} = (a - 1)^n + 1, \text{ } n \in \mathbb{Z}^+.$$

(Note that $\overbrace{a * a * \dots * a}^{5 \text{ times}} = a * a * a * a * a$).

(ii) Hence show that there is exactly one element in T which has finite order, apart from the identity element. Find this element and its order. [10 marks]

Discrete Mathematics

8. [Maximum mark: 30]

- (i) (a) Use the Euclidean algorithm to find the greatest common divisor, d , of 272 and 656. [3 marks]
- (b) Hence express d in the form $272a + 656b$, where $a, b \in \mathbb{Z}$. [3 marks]

(ii) A graph G has adjacency matrix given by

$$\begin{array}{c}
 \text{A} \quad \text{B} \quad \text{C} \quad \text{D} \quad \text{E} \quad \text{F} \\
 \text{A} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ \text{B} \begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 0 \\ \text{C} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ \text{D} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ \text{E} \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 1 \\ \text{F} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}
 \end{array}$$

- (a) Draw the graph G . [2 marks]
- (b) Explain why G has a Eulerian circuit. Find such a circuit. [3 marks]
- (c) Find a Hamiltonian path. [1 mark]

(iii) The terms of the sequence $\{u_n\}$ satisfy the difference equation

$$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = 0, \text{ where } n \in \mathbb{Z}^+ \text{ and } \lambda \in \mathbb{R}.$$

- (a) Show that both $u_n = \lambda^n$ and $u_n = n\lambda^n$ satisfy this difference equation. Hence write down the general solution to this difference equation. [5 marks]
- (b) The terms of the sequence $\{v_n\}$ satisfy the equation

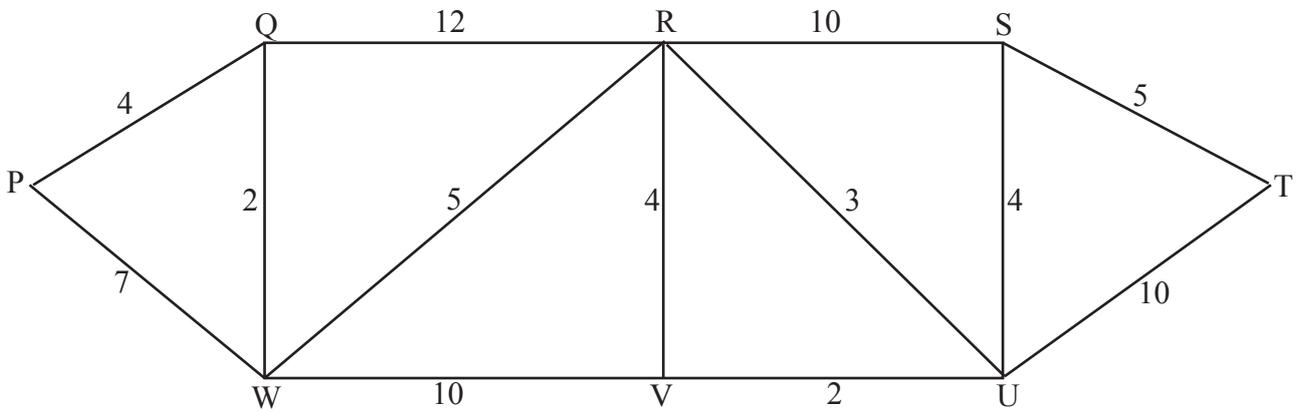
$$v_{n+2} - 6v_{n+1} + 9v_n = 0 \text{ for } n \in \mathbb{Z}^+, \text{ and } v_1 = 2, v_2 = 9.$$

Find an expression for v_n in terms of n . [5 marks]

(This question continues on the following page)

(Question 8 continued)

(iv) The following diagram shows a weighted graph.



Use Dijkstra's Algorithm to find the length of the shortest path between the vertices P and T. Show all the steps used by the algorithm and write down the shortest path.

[8 marks]

Analysis and Approximation

9. [Maximum mark: 30]

(i) Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, and $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, find the first four non-zero terms of the Maclaurin series for $e^{-x} \ln(1+x)$. [6 marks]

(ii) Let $I = \int_2^{3.2} \left(\frac{\ln x}{x^2} \right) dx$.

(a) Use the trapezium rule with **six** sub-intervals to estimate the value of I . [4 marks]

(b) How many sub-intervals are required so that the error in estimating I is less than 10^{-5} ? [8 marks]

(iii) Let $g(x) = \sin^3 x + \cos x$, for $1 \leq x \leq 1.5$.

Use the Newton-Raphson method to find an approximation for a , with an error less than 10^{-8} , such that $g(a)$ is a **maximum**. [7 marks]

(iv) Find the radius of convergence of the series $\sum_{k=1}^{\infty} \frac{(2k-2)!}{k!(k-1)!} x^k$. [5 marks]

Euclidean Geometry and Conic Sections

10. [Maximum mark: 30]

(i) The line $3x + y = 12$ meets the y -axis at P, the x -axis at Q, the line $y - x = 0$ at R and the line $y + x = 0$ at S. Show that P, Q, R, S divide the line segment [PS] in a harmonic ratio. [9 marks]

(ii) The straight line l of gradient m passes through the point $(1, 0)$.

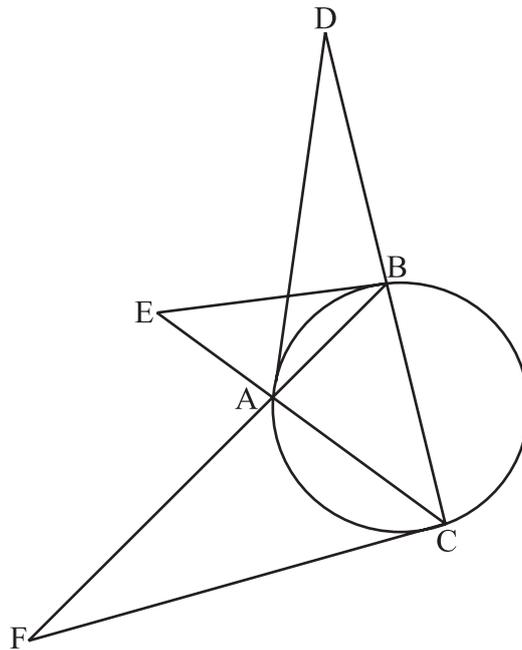
(a) Write down the equation of l . [1 mark]

(b) A parabola has equation $y^2 = 2x$. The line l meets the parabola at U and V. The midpoint of [UV] is denoted by W.

(i) Show that the coordinates of W are $\left(1 + \frac{1}{m^2}, \frac{1}{m}\right)$.

(ii) Show that the locus of W as m varies is a parabola. Find the coordinates of its focus and the equation of its directrix. [10 marks]

(iii)



The figure shows a triangle ABC inside a circle. The tangent to the circle at A meets (BC) at D, the tangent at B meets (CA) at E and the tangent at C meets (AB) at F.

(a) Show that $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. [9 marks]

(b) State briefly, with a reason, what conclusion can be drawn from this result. [1 mark]